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SURVEILLANCE PROBLEMS: A BREAKDOWN MODEL

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ERRATA SHEET

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page 8, first line, $\dfrac{h_1h_2\cdots h_{n-1}}{\dfrac{(t_n-h_n)^2}{(n-1)!}\, e^{\Delta h_n}}$ should read $\dfrac{h_1h_2\cdots h_{n-1}}{\dfrac{(t_n-h_n)^{n-1}}{(n-1)!}\, e^{\Delta h_n}}$.

page 10, line 7 from top, (see assumption 1.b) should read (see assumption 1.c).

page 12, equation (13), $A'(T, w)$ should read $A'(t_b, w)$ wherever it occurs.

page 13, equation (14), $A'(T, w)$ should read $A'(t_b, w)$.

page 16, line 7 from top, $0 < \Delta_2' < \Delta_2$ should read $0 < \Delta_2 < \Delta_2'$.

SURVEILLANCE PROBLEMS: A BREAKDOWN MODEL

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Summary:

An economic model for the surveillance of a production process is proposed and studied in this paper. The state of the production process is described by a Poisson stochastic process, denoted by $x(t)$, with $x=0$ corresponding to the best state. The production process is such that at any instant of time it may go to the next state or may break down. Breakdown can be thought of as a state with a very large negative income. When $x(t) = x$, the income per unit of time is $i(x)$. $i(x)$ is assumed to be a nonincreasing function of x ($x \geq 0$). Repair is the essential part of the model which is decided by observing the production process continuously. It is assumed that the observation can be made without cost and that the result is known immediately. The cost of repair depends on how the process comes to stop. The time between the commencement of operations following repairs and the recurrence of that event is defined as a cycle.

In the first section, the model and the assumptions are discussed in detail. In section 2, some general results are given, the particular cases of which are needed for obtaining the expected length of a cycle and the expected income per cycle. The expression for the long-run average income per unit of time is given in section 4. In section 3, a 'complete class' of strategies, which specify when to stop production and start repair, is defined. It is shown in section 4 that one member of the complete class is optimal.

Two particular cases of $i(x)$ are considered in section 5 and some numerical examples which illustrate the different aspects of the model are included in section 6.

Notation:

1. $x(t) = x$ denotes the state of the production process, where t is the time measured from the beginning of a cycle.
2. $i(x)$ denotes the income per unit of time when $x(t) = x$.
3. $I(x, T)$ denotes the conditional expected income from production in the interval $(t, t+T)$ given that $x(t) = x$ and the production is allowed to continue to $t+T$.
4. $\Delta_1 \geqq 0$, parameter of the Poisson production process.
5. $\Delta_2 \geqq 0$, parameter of the exponential failure (breakdown) distribution.
6. $D = \dfrac{\Delta_1}{\Delta_1+\Delta_2} \leqq 1$.

7a. m_1 denotes the number of time units required to make $x(0) = 0$ when the production is stopped at $x(t) = x$ according to a specified strategy.

7b. k_1 denotes the cost per unit of time doing this work.

7c. m_2, k_2, corresponding numbers for the breakdown situation.

8. t_w denotes the first solution of $x(t) = w+1$.
9. t_b denotes the time when the production process breaks down.
10. R_w denotes the strategy which specifies when to stop production and start repair.
11. $I(R_w)$ denotes the long-run average income per unit of time using the strategy R_w.

1. Introduction:

In an earlier paper, Savage [1962] has proposed a particular model for a production process in which the basic characteristic of the production process is that it tends to wear out unless repairs are made. The times for making repairs are determined by keeping it under surveillance. Two forms of surveillance have been considered; continuous surveillance without cost, and costly surveillance. The production process is assumed to be a Poisson stochastic process, thus allowing the production process to have many states.

A generalization of Savage's model is proposed and studied in this paper. The generalization is in the direction of greater complexity of the model, viz., it allows the machine to break down independent of the production process but preserves all other characteristics of the model. There are situations in which the production process may at any time either go to the next state or breakdown. Breakdown can be thought of as a state with a very large negative income. By introducing the breakdown feature, the field of application is increased. The results are derived only for the case of continuous surveillance without cost.

To be specific, consider a machine which produces output in a continuous stream when it is not in the repair state. The primary interest is in the surveillance of a machine. The problem is then to formulate a strategy which specifies when to stop production and start repairs. The basic block of time is a cycle, the time from beginning production after repair until the recurrence of that event. Let the state of the production process while producing be denoted by $x(t)$, where t is the time measured from the beginning of a cycle. When $x(t) = x$, let the income per unit of time be $i(x)$.

In this paper, several assumptions like the following are made:

1. $x(t)$ is a Poisson process with parameter Δ_1 and $x(0) = 0$. Some of the consequences of this assumption which shall be needed in the discussion are:

   (a). $\text{Prob}(x(t_1)-x(t_2) = x) = \dfrac{e^{-\Delta_1 T}(\Delta_1 T)^x}{x!}$, where $x \geq 0$ and $T = t_1-t_2 \geq 0$.

   (b). $x(t)$ as a function of t, is nondecreasing and with probability one, when $x(t)$ has a point of increase, the size of the increase is one.

   (c). The time between the points of increase of $x(t)$ has an exponential distribution with parameter Δ_1 and mean $\dfrac{1}{\Delta_1}$.

(d). The $x(t)$ process is Markovian.

(e). The $x(t)$ process has stationary independent increaments.

2. The machine breaks down independent of the production process and has an exponential failure distribution with parameter Δ_2.

3. The $x(t)$ process can be observed without cost at all times of production and the observation becomes available immediately.

4. The magnitude of the cost of repair depends on whether the machine breaks down or stops according to a specified strategy. In the former case, the cost of repair can be very high.

5. The income function $i(x)$ is a nonincreasing function of x, the best state is state 0.

Not all the results obtained in this paper require all of these assumptions. Most of the notations used are in accordance with the paper by Savage [1962].

2. Preliminaries:

To proceed with the discussion in detail, one needs the following definitions and results:

Let $I(x, T)$ denote the conditional expected income from production in the interval $(t, t+T)$ given that $x(t) = x$ and production is allowed to continue to $t+T$. In other words

$$(1) \qquad I(x, T) = E[\int_t^{t+T} i(x(s))ds \,|\, x(t) = x] .$$

That $I(x, T)$ does not depend on t follows from the assumption that $x(t)$ is a Poisson process.

Let m_1 denote the number of time units required to make $x(0) = 0$ when the production is stopped at $x(t) = x$ according to a specified strategy and that k_1 denotes the cost per unit of time of doing this work. Let m_2 and

k_2 be the corresponding numbers for the breakdown situation. $(m_2 \geq m_1,\ k_2 \geq k_1)$.

In the following, let $\psi_g(t)$ denote the moment generating function of a random variable x with density function g.

Theorem 1:

Let x and y be two independent random variables, where $\text{Prob}(x < 0) = 0$ and y has an exponential distribution with parameter Δ_2, then

(a) $$E \min(x, y) = \frac{1}{\Delta_2}[1-\psi_g(-\Delta_2)],$$

(b) $$\text{Prob}(x < y) = \psi_g(-\Delta_2) .$$

Proof:

(a). Let

G and g be the cummulative distribution function and density function respectively of x

and

H and h be the cummulative distribution function and density function respectively of y.

Let $z = \min(x, y)$, then

(2) $$F(z) = \text{Prob}[\min(x, y) \leq z] = 1-\text{Prob}[\min(x, y) > z] = 1-[1-G(z)][1-H(z)] .$$

Now

$$Ez = \int_0^\infty z dF(z) = -\int_0^\infty z d\{[1-G(z)]\, e^{-\Delta_2 z}\} .$$

Integrating by parts,

$$Ez = \int_0^\infty [1-G(z)]\, e^{-\Delta_2 z} dz .$$

Again integrating by parts,

(3) $$Ez = \frac{1}{\Delta_2} - \frac{1}{\Delta_2}\int_0^\infty e^{-\Delta_2 z} g(z)dz = \frac{1}{\Delta_2}[1-\psi_g(-\Delta_2)] .$$

(b). $$\mathrm{Prob}(x < y) = \int_0^\infty [\int_0^y g(x)dx] d[1-e^{-\Delta_2 y}] .$$

Integrating by parts,

(4) $$\mathrm{Prob}(x < y) = \int_0^\infty e^{-\Delta_2 y} g(y)dy = \psi_g(-\Delta_2) .$$

Corollary:

In theorem 1, if x is the sum of, say, n independent exponential-random variables (i.e., has a gamma distribution) with parameter Δ_1 then

(5a) $$E \min(x, y) = \frac{1}{\Delta_2}[1-(\frac{\Delta_1}{\Delta_1+\Delta_2})^n] .$$

(5b) $$\mathrm{Prob}(x < y) = (\frac{\Delta_1}{\Delta_1+\Delta_2})^n .$$

Proof:

The moment generating function of a gamma-random variable with parameter Δ_1 evaluated at $-\Delta_2$ is (5b).

Definitions:

Let t_i be the time of the $i^{\underline{th}}$ jump of a Poisson process $x(t)$ with parameter Δ. Let y be a non-negative random variable independent of the Poisson process. Let n be a fixed positive integer. Let $z = \min(y, t_n)$. Let r be the state of the process just before z so that $t_r < z \leq t_{r+1}$.

Theorem 2:

Given r and z, the random variables $t_1, t_2, \ldots, t_r$ have the distribution of the order statistics in a sample of size r from a uniform distribution in the interval $(0, z)$.

Proof:

The proof consists of two parts.

(a). Let $t_n < y$, then

$$\text{Prob}[t_1, t_2, \ldots, t_{n-1} | t_n,\ t_n < y] = \frac{\int_{t_n}^{\infty} f(t_1, t_2, \ldots, t_{n-1},\ t_n,\ y)dy}{\int_{t_n}^{\infty} f(t_n,\ y)dy}$$

(since y is independent of t's)

$$= \frac{f(t_1, t_2, \ldots, t_n)}{f(t_n)}$$

$$= f(t_1, t_2, \ldots, t_{n-1} | t_n) .$$

Now the rest is the proof of one of the exercises given by Parzen [1963, p. 143]. Let the first event occur in $(s_1,\ s_1+h_1)$, the second event in $(s_2,\ s_2+h_2)$,..., the $(n-1)^{\underline{th}}$ event in $(s_{n-1},\ s_{n-1}+h_{n-1})$ and the $n^{\underline{th}}$ event in $(t_n,\ t_n+h_n)$. (The intervals are non-overlaping.) Then

$$f(s_1, s_2, \ldots, s_{n-1} | t_n) h_1 h_2 \ldots h_{n-1}$$

$$= \text{Prob}[s_1 \leqq t_1 \leqq s_1+h_1,\ s_2 \leqq t_2 \leqq s_2+h_2, \ldots, s_{n-1} \leqq t_{n-1} \leqq s_{n-1}+h_{n-1} | t_n]$$

$$= \frac{(\Delta h_1 e^{-\Delta h_1} \ldots \Delta h_{n-1} e^{-\Delta h_{n-1}})\ \Delta h_n e^{-\Delta h_n}\ e^{-\Delta(t_n - h_1 - \ldots - h_{n-1})}}{\frac{[\Delta(t_n - h_n)]^{n-1}}{(n-1)!}\ e^{-\Delta(t_n - h_n)}\ \Delta h_n e^{-\Delta h_n}}$$

$$= \frac{h_1 h_2 \ldots h_{n-1}}{\frac{(t_n - h_n)^2}{(n-1)!} e^{\Delta h_n}} . \qquad \text{Now as } h_n \to 0$$

$$= \frac{(n-1)! h_1 h_2 \ldots h_{n-1}}{(t_n)^{n-1}} .$$

This is the distribution of ordered t's in a sample of size $n-1$ from a uniform distribution in the interval $(0, t_n)$.

(b). Let $y < t_{r+1}$, then

$$\text{Prob}[t_1, t_2, \ldots, t_r | r, y, y < t_{r+1}] = \frac{\int_y^\infty f(t_1, t_2, \ldots, t_r, t_{r+1}, y) dt_{r+1}}{\int_y^\infty f(t_{r+1}, y) dt_{r+1}}$$

(since y is independent of t's)

$$= \frac{\int_y^\infty f(t_1, t_2, \ldots, t_r, t_{r+1}) dt_{r+1}}{\int_y^\infty f(t_{r+1}) dt_{r+1}} .$$

Let $t_{k+1} - t_k$ be substituted for t_{k+1}, $k=0,1,2,\ldots,r$; $t_0 = 0$, and since $t_1, t_2 - t_1, \ldots, t_{r+1} - t_r$ are independent and exponentially distributed random variables, then

$$\text{Prob}[t_1, t_2, \ldots, t_r | r, y, y < t_{r+1}] = \frac{\int_{y-t_r}^\infty h(t_1) h(t_2 - t_1) \ldots h(t_{r+1} - t_r) d(t_{r+1} - t_r)}{\int_{y-t_r}^\infty h(t_{r+1} - t_r) d(t_{r+1} - t_r)}$$

$$= \frac{h(t_1) h(t_2 - t_1) \ldots h(t_r - t_{r-1}) e^{-\Delta(y - t_r)}}{e^{-\Delta(y - t_r)}} ,$$

(changing $t_{k+1}-t_k$ to t_{k+1}, $k=0,1,2,\ldots,r-1$; $t_0 = 0$),

$$= \frac{f(t_1,t_2,\ldots,t_r, y)}{g(y)} = f(t_1,t_2,\ldots,t_r|r, y).$$

Now Parzen [1963, p. 140] has shown that given $x(T) = r$, where $t_1 \leqq t_2 \leqq\ldots\leqq t_r \leqq T$ (T some fixed number), then $t_1,t_2,\ldots,t_r$ have the distribution of the order statistics in a sample of size r from a uniform distribution in the interval $(0, T)$.

In the case (b)., y is a random variable, but for each value of y, the above theorem of Parzen is true, hence true for the random variable y. Combining case (a). and (b)., theorem 2 holds.

Now by virtue of this theorem, the density function of ordered t_k is

$$(6) \qquad f(t_k) = \frac{r!}{(k-1)!(r-k)!} \left(\frac{t_k}{z} \right)^{k-1} \left[1- \frac{t_k}{z} \right]^{r-k} \frac{1}{z}, \qquad \begin{array}{l} 0 \leqq t_k \leqq z \\ k=1,2,\ldots,r \end{array} .$$

Let $T_k = \frac{t_k}{z}$, then

$$f(T_k) = \frac{r!}{(k-1)!(r-k)!} T_k^{k-1}(1-T_k)^{r-k} \qquad 0 \leqq T_k \leqq 1$$

and

$$(7) \qquad ET_k = \frac{k}{r+1} .$$

3. Procedures:

Consider the strategies R_w according to which the production process is to be placed in the repair state at the first instant that $x(t) = w+1$, $w \geqq 0$. The class of strategies R_w is a complete class. According to the assumption $x(t)$ is a Poisson process and has Markoff properties. Therefore, the decisions depend only on the present state and not on the past history of the process.

Also note that the future history of the process does not depend on how long the process has spent in a given production state. So if it is ever desirable to place the process in repair when in a particular state, it should be as soon as the process enters that state.

Let t_w be the first solution of $x(t) = w+1$. Also, let t_b be the time when the machine breaks down. Then t_w and t_b are random variables. t_w has a gamma distribution with parameter Δ_1 (see assumption 1.b) and t_b has an exponential distribution with parameter Δ_2 (see assumption 2). One can also note from the definition of a cycle that the length of a cycle is also a random variable.

Now using the strategy R_w,

(a) the expected length of a cycle is

$$E \min(t_w, t_b) + m_1 \text{Prob}(t_w < t_b) + m_2 \text{Prob}(t_b < t_w).$$

(b) the expected income per cycle is

$$EI[0, \min(t_w, t_b)] - m_1 k_1 \text{Prob}(t_w < t_b) - m_2 k_2 \text{Prob}(t_b < t_w).$$

Then, it is shown by Girshick and Rubin [1952] that with probability one, the long-run average income per unit of time denoted as $I(R_w)$ is

(c) $$I(R_w) = \frac{EI[0, \min(t_w, t_b)] - m_1 k_1 \text{Prob}(t_w < t_b) - m_2 k_2 \text{Prob}(t_b < t_w)}{E \min(t_w, t_b) + m_1 \text{Prob}(t_w < t_b) + m_2 \text{Prob}(t_b < t_w)} .$$

The objective is to maximize the long-run average income per unit of time. This can be done by first evaluating $I(R_w)$ and then finding the maximizing value w^* of w.

4. Computations:

(a) The expected length of a cycle is

$$E \min(t_w, t_b) + m_1 \text{Prob}(t_w < t_b) + m_2 \text{Prob}(t_b < t_w).$$

Using theorem 1, this expression is

$$\frac{1}{\Delta_2}[1-D^{w+1}] + D^{w+1}(m_1-m_2) + m_2 , \tag{8}$$

where $D = \left(\frac{\Delta_1}{\Delta_1+\Delta_2} \right)$.

(b) The expected income per cycle (without the cost of repair) is

$$EI[0, \min(t_w, t_b)] = E\left[\int_0^{\min(t_w, t_b)} i(x(s))ds \right]. \tag{9}$$

Expectation is over both $\min(t_w, t_b)$ and $x(s)$. This expression can also be written as

$$\text{Prob}(t_w < t_b)\, E_{t_w}\left[\int_0^{t_w} E\{i(x(s))|t_w, t_w < t_b\}ds \right]/t_w < t_b \tag{10}$$

$$+ \text{Prob}(t_b < t_w)\, E_{t_b}\left[\int_0^{t_b} E\{i(x(s))|t_b, t_b < t_w\}ds \right]/t_b < t_w .$$

Now consider first

$$\int_0^{t_w} E\{i(x(s))|t_w, t_w < t_b\}ds = E \int_0^{t_w} \{i(x(s))|t_w, t_w < t_b\}ds$$

$$= E\left\{ \int_0^{t_0} i(x(s))ds + \int_{t_0}^{t_1} i(x(s))ds + \ldots + \int_{t_{w-1}}^{t_w} i(x(s))ds \right\}/t_w, t_w < t_b$$

$$= E\{t_0 i(0) + (t_1-t_0)i(1) + \ldots + (t_w-t_{w-1})i(w)\}/t_w, t_w < t_b .$$

(using theorem 2),

$$= t_w \sum_{h=0}^{w} \frac{i(h)}{w+1} = t_w A(w) \text{ say.}$$

$\therefore$ the first term of (10) is

(11) $$\text{Prob}(t_w < t_b)\ E_{t_w}[t_w A(w)]/t_w < t_b$$

$$= \int_0^\infty \left[\int_0^{t_b} t_w A(w) g(t_w) dt_w \right] h(t_b) dt_b \ = \ \frac{1}{\Delta_1} D^{w+2} \sum_{h=0}^{w} i(h) \ .$$

Now consider, $\int_0^{t_b} E\{i(x(s)) | t_b,\ t_b < t_w\} ds$.

Let the machine break down between h and $(h+1)^{\underline{st}}$ state. Then $t_b < t_w \iff h+1 < w+1$. For convenience let $T = t_b$, then $x(T) = h$. Write the above expression as

(12) $$\int_0^T E_{h/T}\, E_{x(s)/h,T}\{i(x(s)) | h,\ h+1 < w+1\} ds \ .$$

Using theorem 2, one gets as before

$$E_{x(s)/h,T} \int_0^T \{i(x(s)) | h,\ h+1 < w+1\} ds \ = \ T \sum_{j=0}^{h} \frac{i(j)}{h+1} \ .$$

Then, (12) can be expressed in the following form:

$$\frac{T \displaystyle\sum_{h=0}^{w} \frac{e^{-\Delta_1 T} (\Delta_1 T)^h}{h!} \sum_{j=0}^{h} \frac{i(j)}{h+1}}{\displaystyle\sum_{h=0}^{w} \frac{e^{-\Delta_1 T} (\Delta_1 T)^h}{h!}} \ = \ TA'(T, w) \text{ say.}$$

After replacing T by t_b, the second term of (10) is

(13) $$\text{Prob}(t_b < t_w)\ E_{t_b}[t_b A'(T, w)]/t_b < t_w = \int_0^\infty \int_{t_b}^\infty t_b A'(T, w) h(t_b) g(t_w) dt_b dt_w \ .$$

Integrating over t_w first and using the result that

$$\int_{t_b}^{\infty} g(t_w)dt_w = \sum_{h=0}^{w} \frac{e^{-\Delta_1 t_b}(\Delta_1 t_b)^h}{h!} \text{ , we can now write equation (13) as}$$

$$(14) \qquad \text{Prob}(t_b < t_w)\ E_{t_b}[t_b A'(T, w)]/t_b < t_w = \frac{\Delta_2}{\Delta_1^2} \sum_{h=0}^{w} \sum_{j=0}^{h} i(j)D^{h+2} \text{ .}$$

The right hand side of (14) can also be written as

$$\frac{\Delta_2}{\Delta_1^2} \sum_{j=0}^{w} \sum_{h=j}^{w} i(j)D^{h+2} \text{ , which reduces to}$$

$$(15) \qquad \frac{1}{\Delta_1} \sum_{h=0}^{w} i(h)D^{h+1} - \frac{1}{\Delta_1} D^{w+2} \sum_{h=0}^{w} i(h) \text{ .}$$

Combining (11) and (14) in (9), (9) becomes

$$EI[0, \min(t_w, t_b)] = \frac{1}{\Delta_1} \sum_{h=0}^{w} i(h)D^{h+1} \text{ .}$$

Hence,

(c) The long-run average income per unit of time is

$$(16) \qquad I(R_w) = \frac{\frac{1}{\Delta_1} \sum_{h=0}^{w} i(h)D^{h+1} + D^{w+1}(m_2k_2 - m_1k_1) - m_2k_2}{\frac{1}{\Delta_2}[1 - D^{w+1}] + D^{w+1}(m_1 - m_2) + m_2} \text{ .}$$

To use $I(R_w)$ correctly, one has to notice that w is a non-negative integer; Δ_1, Δ_2 are dimensionally $(\text{time})^{-1}$; $i(x)$, k_1, k_2 are dimensionally (money/time) and m_1, m_2 are dimensionally (time).

The next problem is to find an optimizing value w^*. However, (16) is very involved in w. Hence, it will not be easy to obtain a convenient expression for

w^*. But it is important to investigate whether there exists a unique w^* or not. The following theorem does that.

<u>Theorem 3:</u>

A sufficient conditon for $I(R_w)$ to be unimodal is that $i(h+1) - i(h) < 0$, $h \geq 0$.

<u>Proof:</u>

$$I(R_w) = \frac{\Delta_2 \sum_{h=0}^{w} i(h)D^{h+1} + D^{w+1}\Delta_1\Delta_2(m_2k_2-m_1k_1) - m_2k_2\Delta_1\Delta_2}{\Delta_1(1+m_2\Delta_2) + D^{w+1}\Delta_1[(m_1-m_2)\Delta_2-1]}$$

Let

$$P = \Delta_1\Delta_2(m_2k_2-m_1k_1) \quad , \qquad M = m_2k_2\Delta_1\Delta_2 \quad ,$$

$$T = \Delta_1(1+m_2\Delta_2) \quad , \qquad S = \Delta_1[(m_1-m_2)\Delta_2-1] \; .$$

Consider

$$I(R_{w+1})-I(R_w) = \frac{\Delta_2 \sum_{h=0}^{w+1} i(h)D^{h+1} + D^{w+2}P-M}{T+D^{w+2}S} - \frac{\Delta_2 \sum_{h=0}^{w} i(h)D^{h+1} + D^{w+1}P-M}{T+D^{w+1}S}$$

$$= [(T+D^{w+1}S)(T+D^{w+2}S)]^{-1} f(w) \quad \text{say} .$$

Since the term inside the square bracket is positive, the sign of $I(R_{w+1})-I(R_w)$ depends on $f(w)$, where

$$f(w) = T\Delta_2[\sum_{h=0}^{w+1} i(h)D^{h+1} - \sum_{h=0}^{w} i(h)D^{h+1}] - TPD^{w+1}(1-D)$$

$$+ D^{w+1}\Delta_2 S[\sum_{h=0}^{w+1} i(h)D^{h+1} - D\sum_{h=0}^{w} i(h)D^{h+1}] - MD^{w+1}S(1-D)$$

$$= T\Delta_2 D^{w+2} i(w+1) - TPD^{w+2}\frac{\Delta_2}{\Delta_1} + D^{w+1}\Delta_2 S[D^{w+2}i(w+1) + (1-D)\sum_{h=0}^{w} i(h)D^{h+1}]$$

$$- MSD^{w+2}\frac{\Delta_2}{\Delta_1}$$

$$f(w) = \Delta_2 D^{w+2} i(w+1)[T+D^{w+1}S] + D^{w+3}\frac{\Delta_2^2}{\Delta_1} S \sum_{h=0}^{w} i(h)D^h - D^{w+2}\frac{\Delta_2}{\Delta_1}[TP+MS] .$$

Now

$$T+D^{w+1}S = (T+S)-S(1-D^{w+1}) = (T+S)-S\left(\frac{1-D^{w+1}}{(1-D)}\right)(1-D) = (T+S)-S(1-D)\sum_{h=0}^{w} D^h$$

$$\therefore f(w) = D^{w+2}\Delta_2[(T+S)i(w+1) - \frac{1}{\Delta_1}(TP+MS)] - D^{w+3}\frac{\Delta_2^2}{\Delta_1} S \sum_{h=0}^{w} D^h[i(w+1)-i(h)] .$$

Now putting back the values of P, M, S, T in the first term,

$$f(w) = D^{w+2} m_1\Delta_1\Delta_2^2[i(w+1)+k_1-m_2\Delta_2(k_2-k_1)] - D^{w+3}\frac{\Delta_2^2}{\Delta_1} S \sum_{h=0}^{w} D^h[i(w+1)-i(h)] .$$

Now if $i(h+1)-i(h) < 0$, $h \geq 0$, then the second term on the right is always negative ($S < 0$). One can also notice that as w increases the first term on the right decreases. Hence $f(w)$ is a decreasing function of w. This implies that there is at most one change of sign as w increases, i.e., there exists a unique $w^* \geq 0$ which maximizes $I(R_w)$.

Some of the qualitative properties of $I(R_w)$ are the following:

(a). When $\Delta_2 = 0$, the model reduces to that of Savage [1962].

<u>Definition:</u> A random process x dominates the random process x', if and only if there exists a function $g(x)$ defined on the range of x such that $g(x)$ has the same distribution as x'.

<u>Lemma:</u> In any decision problem, if x dominates x', than any strategy based on x' when x' is being observed can also be used when x is observed.

<u>Lemma:</u> In the breakdown model, if one considers $\Delta_2 < \Delta_2'$, then the observed process with Δ_2' is dominated by the observed process with Δ_2.

(b). **Theorem 4:**

The maximum long-run average income per unit of time is a decreasing function of Δ_2.

Proof:

Consider two models which differ only in the value of the breakdown parameter. For the first model assume this parameter has the value Δ_2 and for the second model Δ_2'. We assume $0 < \Delta_2' < \Delta_2$. It will be shown that the maximum income from the Δ_2 model is greater than the maximum income from the Δ_2' model.

By introducing randomization one can generate a new stochastic process, denoted by x^*, associated with the Δ_2-process such that

(i) x^* has the same distribution as the Δ_2'-process $(\Delta_2 < \Delta_2')$

(ii) the occurence of the events in the x^*-process will not be later than the events in the Δ_2-process.

Hence, if repairs are made according to the optimal strategy associated with the Δ_2'-process when in fact observing the x^*-process, the distribution of the lengths of cycles will be exactly the same. The income from production during the cycles will also be exactly the same. But the costs of repairs with x^*-process will be less than those associated with the Δ_2'-process. This is because the costly repairs will always be made if the Δ_2'-process stops but the costly repairs will not be made all of the time when the x^*-process stops. Thus using the strategy on x^* or equivalently on Δ_2-process the income from Δ_2-process can be made to exceed the average income per unit of time from the Δ_2'-process.

Finally, the x^*-process can be constructed as follows: Let t_b be the random variable of breakdown times for the Δ_2-process. Let t^* be an exponential random variable, generated by using a table of random numbers, with parameter equal to $\Delta_2'-\Delta_2$. Then it is clear that $x^* = \min(t_b, t^*)$ has the desired properties (i) and (ii).

(c). <u>Conjecture:</u>

w^* is an increasing function of Δ_2.

In the following, two particular cases of $i(x)$ are considered.

5. <u>Particular cases:</u>

(a). Consider $i(x) = A, \quad x \geq 0, \quad A > 0.$

Though this case is not important from the point of view of its practical usefulness, still it is considered here as an example.

$I(R_w)$ can be obtained either from (16) or directly using (8) as

$$(17) \qquad I(R_w) = \frac{\frac{A}{\Delta_2}(1-D^{w+1}) + D^{w+1}(m_2k_2-m_1k_1) - m_2k_2}{\frac{1}{\Delta_2}(1-D^{w+1}) + D^{w+1}(m_1-m_2) + m_2} .$$

Notice that $w^* = +\infty$ (i.e., continue the machine until it breaks down) is the optimizing value, and

$$(18) \qquad I(R_{w^*}) = \frac{A-m_2k_2\Delta_2}{1+m_2\Delta_2} .$$

(b). Consider $i(x) = A-Bx, \quad x \geq 0, \quad B > 0, \quad A \geq 0.$

From (16),

$$(19) \qquad I(R_w) = \frac{D^{w+1}[B(w+1)+B\frac{\Delta_1}{\Delta_2} - A+\Delta_2(m_2k_2-m_1k_1)]+A-B\frac{\Delta_1}{\Delta_2} - m_2k_2\Delta_2}{(1+m_2\Delta_2)-D^{w+1}[1+\Delta_2(m_2-m_1)]} .$$

To see that for $\Delta_2 = 0$, this expression reduces to that of Savage [1962], divide both numerator and denominator by Δ_2 and then combine the terms.

Since $i(k+1)-i(k) < 0$, theorem 3 says that there exists a unique w^* which maximizes (19). To work with the critical equation associated with (19), the computations will be made as if w is a continuous variable. Then setting the derivative of (19) with respect to w equal to zero, one obtains

$$B[1+\Delta_2(m_2-m_1)]D^{w+1}-wB(1+m_2\Delta_2)\log D-B(1+m_2\Delta_2) - [B+B(m_2\Delta_2+m_1\Delta_1)-m_1\Delta_2(A+k_1)+m_1m_2\Delta_2^2(k_2-k_1)]\log D = 0 . \tag{20}$$

Let w^* be the root of (20), then it can be shown that one of the integers adjacent to w^* is the optimizing value. To get $I(R_{w^*})$, i.e., the maximum long-run average income per unit of time, one can use either (19) to get an exact value, or the following.

Let

$$I(R_w) = \frac{H(w)}{G(w)} , \tag{21}$$

then it is easy to see that

$$I(R_{w^*}) \doteq \frac{H'(w^*)}{G'(w^*)} , \tag{22}$$

where prime denotes the derivative with respect to w evaluated at w^*.

$$\therefore I(R_{w^*}) = \frac{B+[B(w^*+1)+B\dfrac{\Delta_1}{\Delta_2} -A+\Delta_2(m_2k_2-m_1k_1)]\log D}{-[1+\Delta_2(m_2-m_1)]\log D} . \tag{23}$$

In working with (23), the root of (20) should be used to get a close approximation.

6. Examples:

Consider $i(x) = A-Bx$, $x \geqq 0$, $B > 0$, $A \geqq 0$.

$I(R_w)$ is given by (19) and the critical equation associated with it is given by (20). Notice that while solving (20) for w, one gets a fractional valued w^*. As remarked earlier, the correct value of w^* is one of the adjacent integers. The following table indicates the effect of change in the values of the parameters on w^* and $I(R_{w^*})$.

Example	A	B	m_1	m_2	k_1	k_2	Δ_1	Δ_2	w^*	$I(R_{w^*})$
1	10	1	2	2	2	2	1	1	8	1.66
2	10	1	2	4	2	2	1	1	5	0.2
3	10	1	2	2	2	4	1	1	5	0.34
4	10	1	2	2	2	4	9	1	6	-0.88
5	10	1	2	2	2	2	9	1	9	0.053
6	10	1	2	0	2	0	1	0	4	5.5
7	10	1	2	2	2	2	1	9	10	-1.37
8	0	1	2	2	2	2	1	1	1	-1.55

Examples 2 and 3 illustrate the effect of increase in the cost of repair after breakdown. Also notice examples 4 and 5. Comparing examples 1, 6 and 7, one can see that it illustrates theorem 4, viz., $I(R_{w^*})$ is a nonincreasing function of Δ_2 and the conjecture that w^* is a nondecreasing function of Δ_2.

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